

# INSCRIBING A REGULAR OCTAHEDRON INTO POLYTOPES

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**ABSTRACT.** We prove that any simple polytope (and some non-simple polytopes) in  $\mathbb{R}^3$  admits an inscribed regular octahedron.

## 1. INTRODUCTION

The famous theorem of Schnirelmann asserts that for every closed simple piece-wise smooth curve  $\gamma$  there exists a square  $Q$  such that all four vertices of  $Q$  are on  $\gamma$ .

In the thesis of Vladimir Makeev [2] the following theorem was proved (reproved and generalized for higher prime power dimensions in [4, 6]):

**Theorem 1.1.** *Let  $H \subset \mathbb{R}^3$  be an image of some smooth embedding of  $S^2$ . Let  $C$  be some  $\mathbb{Z}_3$ -symmetric octahedron. Then there exists an octahedron  $C' \subset \mathbb{R}^3$  similar to  $C$  with all its vertices lying on  $H$ .*

*Remark 1.2.* The word *similar* here means equivalent up to a similarity transform with positive determinant. By *inscribing* a polytope into a surface  $H$  we will always mean finding its similar copy such that all its vertices lie on  $H$ .

It is known (see [7, 3, 9] for example) that squares in the plane can be also inscribed into any polygonal simple curve; the approximation by smooth curves and going to the limit works well in this case. The key feature here is that if you look at the square from some direction in the plane then you do not see one of its vertices. The problem of inscribing a square into any continuous simple closed curve in the plane remains unsolved, the approximation by smooth curves seems to be insufficient in this case, see [3] for further information about this.

In  $\mathbb{R}^3$  the situation is different even for regular octahedra: One can see all the vertices of an octahedron from some directions. Thus we have to be careful when going to the limit and this is the main content of this paper. Note that in the plane there exist direct proofs [8] of the Schnirelmann theorem for polygonal curves, while in this paper we cannot avoid using the smooth case and going to the limit.

The main result of this paper is:

**Theorem 1.3.** *Suppose  $P$  is a simple polytope in  $\mathbb{R}^3$ . Then there exists a regular octahedron inscribed into  $\partial P$ .*

*Remark 1.4.* A weaker result for nonsimple polytopes is Theorem 4.4 in Section 4.

*Remark 1.5.* We only prove this theorem for inscribing a regular octahedron. The case of any  $\mathbb{Z}_3$ -symmetric octahedron (as in [4, 5]) therefore remains open.

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## 2. APPROXIMATION OF $\partial P$ BY SMOOTH SURFACES

We are going to use the following way to approximate a polytope by smooth bodies:

**Definition 2.1.** Let  $P_\varepsilon$  be the union of all  $\varepsilon$ -balls that are contained in  $P$

$$P_\varepsilon = \bigcup_{B_\varepsilon(x) \subseteq P} B_\varepsilon.$$

The body  $P_\varepsilon$  has a smooth boundary and admits an inscribed regular octahedron. Moreover in [4, 5] it is proved that there is a nontrivial  $\mathbb{Z}_3$ -equivariant 1-homology class (in modulo 3 homology) of such octahedra in the configuration space of all octahedra, which is naturally isomorphic to the space of similarity transforms with positive determinant  $S_3 = \mathbb{R}^+ \times \mathbb{R}^3 \times SO(3)$ . The proof in [4, 5] actually was in terms of some relative cohomology, which is actually the same as the codimension 1 compact support cohomology of  $S_3/\mathbb{Z}_3$ , the latter being Poincaré dual to the 1-dimensional homology of  $S_3/\mathbb{Z}_3$  (see [1]). From here on we use 1-dimensional homology cycles understood intuitively, but everything becomes rigorous when passing to codimension 1 compact support cohomology of  $S_3/\mathbb{Z}_3$ .

The group  $\mathbb{Z}_3 \subset SO(3)$  here permutes the coordinates and corresponds to cyclic permutations of the three axes of the octahedron.

Now let  $\varepsilon$  tend to zero. If the diameters of the inscribed octahedra of  $P_\varepsilon$  do not tend to zero, then we obtain an inscribed octahedron for  $P$  by the standard compactness considerations. Assume the contrary: the maximum diameter of inscribed octahedra for  $P_\varepsilon$  is at most  $\delta(\varepsilon)$  and  $\lim_{\varepsilon \rightarrow +0} \delta(\varepsilon) = 0$ . Denote the set of inscribed octahedra for  $P_\varepsilon$  by  $\mathcal{I}_\varepsilon \in S_3$ .

Since  $\delta(\varepsilon)$  tends to zero the octahedra from  $\mathcal{I}_\varepsilon$  tend (say, in Hausdorff metric) to points at the boundary of  $P$ . First observation is that they obviously cannot tend to a relative interior point of a facet. Moreover, the detailed analysis near an edge shows that for small enough  $\varepsilon$  the octahedra in  $\mathcal{I}_\varepsilon$  cannot tend to an interior point of an edge. Indeed, if we project  $\partial P$  along an edge then we obtain a plane angle  $A$ , the smoothening  $\partial P_\varepsilon$  being projected to a smoothened plane angle  $A_\varepsilon$ . Let  $C$  be an octahedron inscribed into  $P_\varepsilon$ ; it projects to a quadrilateral or hexagon  $C'$  inscribed into  $A_\varepsilon$ . Now it remains to note that  $C'$  is centrally symmetric and we cannot see one of its vertices from any direction (of course, the invisible vertex depends on the direction), while we can see the entire  $A_\varepsilon$  from some directions. This is a contradiction and we conclude:

**Lemma 2.2.** *If there is no inscribed octahedron in  $P$  then all octahedra in  $\mathcal{I}_\varepsilon$  tend to vertices of  $P$  in Hausdorff metric. For small enough  $\varepsilon$  the family  $\mathcal{I}_\varepsilon$  becomes a disjoint union of the sets  $\mathcal{I}_\varepsilon(v)$ , corresponding to different vertices  $v \in P$ .*

The decomposition of  $\mathcal{I}_\varepsilon$  into the sets  $\mathcal{I}_\varepsilon(v)$  corresponds to decomposition of the 1-homology class of inscribed octahedra into a smooth body into a sum of 1-homology classes. Hence we have proved:

**Lemma 2.3.** *There exists a vertex  $v$ , such that for arbitrarily small  $\varepsilon$  the octahedra  $\mathcal{I}_\varepsilon(v)$  (inscribed into  $P_\varepsilon$  near  $v$ ) carry a nontrivial  $\mathbb{Z}_3$ -equivariant 1-homology.*

Thus we have to study the situation near the vertices of  $P$ . More precisely, we have to consider solid angles  $A(v)$  of corresponding vertices of  $P$  and their smoothenings  $A_\varepsilon(v)$  (not depending on  $\varepsilon > 0$  essentially because of a possible homothety).

We want to describe the  $\mathbb{Z}_3$ -equivariant 1-homology of the set of octahedra inscribed into every  $A_\varepsilon(v)$ . The configuration space  $S_3$  of all octahedra is homotopy equivalent to  $SO(3)$  and it has trivial homology modulo 3 in dimensions 1 and 2; the  $\mathbb{Z}_3$ -equivariant 1-homology modulo 3 (the homology of  $S_3/\mathbb{Z}_3$ ) is therefore equal to  $H_1(\mathbb{Z}_3; \mathbb{F}_3) = \mathbb{F}_3$ . We choose the generator of  $H_1(S_3/\mathbb{Z}_3; \mathbb{F}_3)$  to be equal to the 1-homology of octahedra inscribed into a generic smooth convex body.

The homology class of inscribed octahedra in  $A_\varepsilon(v)$  is well defined in the case when there are no arbitrarily large octahedra inscribed in  $A_\varepsilon$  (note that we actually work in compact support cohomology and therefore must ensure that everything remains bounded); in this case it corresponds to the well defined 1-homology class. Using an appropriate homothety we conclude that fixed  $\varepsilon$  and arbitrarily large octahedron is the same as arbitrarily small  $\varepsilon$  and a fixed size octahedron. By compactness considerations we conclude the following:

**Lemma 2.4.** *If  $A_\varepsilon(v)$  has no well defined homology class of inscribed octahedra then there exists an octahedron inscribed into the original angle  $A(v)$ .*

**Definition 2.5.** Call a solid angle  $A$  *special* if it admits an inscribed octahedron.

Denote the configuration space of all congruence classes ( $SO(3)$ -orbits) of solid angles (with  $n \geq 3$  facets) by  $\mathcal{A}^n$ , and denote the subset of special angles  $\mathcal{A}_S^n$ , this set is obviously closed. Define the function

$$\varphi : \mathcal{A}^n \setminus \mathcal{A}_S^n \rightarrow \mathbb{F}_3$$

by assigning to a solid angle  $A$  the 1-homology of octahedra (divided by the generator of  $H_1(S_3/\mathbb{Z}_3; \mathbb{F}_3)$ ) inscribed into the smoothened solid angle  $A_\varepsilon$ .

**Lemma 2.6.** *The function  $\varphi$  is locally constant on  $\mathcal{A}^n \setminus \mathcal{A}_S^n$ .*

*Proof.* Fix some  $\varepsilon > 0$ . Under deformations of a solid angle  $A$  that avoid arbitrarily large inscribed octahedra for its smoothening  $A_\varepsilon$ , the smoothenings  $A_\varepsilon$  are deformed continuously and the sizes of their inscribed octahedra are locally bounded. Hence by the continuity of the 1-homology (or compact support cohomology) the class of inscribed into  $A_\varepsilon$  octahedra is preserved.  $\square$

Now in the inscribing problem we have the following options:

- a)  $P$  has a special angle. In this case it already has an inscribed octahedron.
- b) If (the sum is over all vertices in  $P$ )

$$(2.1) \quad \sum_{v \in P} \varphi(A(v)) \neq 1$$

then  $P$  admits an inscribed octahedron. This follows from summing up the homology classes.

Note that for *non-angular* solid angles  $A$  (the definition of *non-angular* is given in [4]) there are no inscribed octahedra and the value  $\varphi(A)$  is zero. In particular, every solid angle close enough to a halfspace has  $\varphi(A) = 0$ .

### 3. PROOF OF THEOREM 1.3

If the polytope  $P$  is simple then we deal with  $\mathcal{A}_S^3 \subset \mathcal{A}^3$ . Taking into account the observations in the previous section, we see that to prove Theorem 1.3 it is enough to prove the following lemma (because in this case the left hand part of (2.1) vanishes):

**Lemma 3.1.** *The set  $\mathcal{A}^3 \setminus \mathcal{A}_S^3$  is arcwise connected.*

The proof of Lemma 3.1 will follow from the description of all solid angles that admit an inscribed octahedron:

**Lemma 3.2.** *A solid angle  $A$  is circumscribed around an octahedron if and only if it is possible to place its correspondent spherical triangle  $v_1v_2v_3$  in the interior of the regular spherical triangle  $t_1t_2t_3$  with  $|t_1t_2| = \pi/3$  in a following way: the vertices  $v_1$  and  $t_1$  coincide, the vertex  $v_2$  lies on the segment  $t_1t_2$ , and  $v_3$  lies inside the triangle  $t_1v_2t_3$ .*

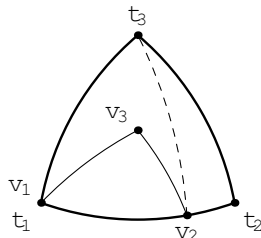


Fig. 1.

*Proof.* Let us see how an octahedron  $C$  could be inscribed into  $A$ . There are two alternatives:

Case 1: Some three vertices of  $C$  are on one facet of  $A$ , two are on the other facet, and one is on the third facet.

Case 2: Every facet of  $A$  contains two vertices of  $C$ .

There are other degenerate cases, but they all are limit cases of these two cases.

Denote the vertices of the octahedron by  $a, b, c, a', b',$  and  $c'$  (see Figure 2).

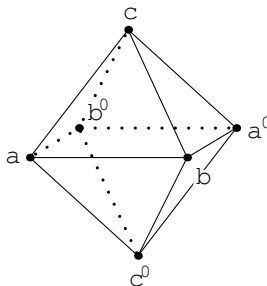


Fig. 2.

Consider the first case. Suppose one of the facets of the angle  $A$  contains the facet  $a'b'c'$ , the second facet contains the edge  $ab$ , and the third facet contains the vertex  $c$ .

Denote the vertex of the angle  $A$  by  $v$ . It is easy to see that the common edge of the first and the second facets of  $A$  is parallel to the edge  $ab$  of  $C$ , because both facets are parallel to  $ab$ . Without loss of generality we may assume that the point  $b$  is closer to  $v$  than  $a$ . Let  $v_1$  be the vertex of the spherical triangle that corresponds to the common edge of the first and the second facet of  $A$ ,  $v_2$  correspond to the third and the second facet, and  $v_3$  is the remaining vertex (see Figure 3).

Let  $t_1, t_2,$  and  $t_3$  be the vertices of the regular spherical triangle that corresponds to the vectors  $\vec{ba}, \vec{c'b'},$  and  $\vec{a'c}$ .

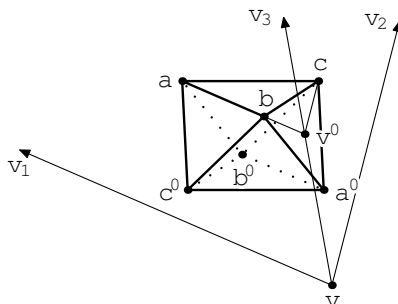


Fig. 3.

Denote by  $v'$  the point where the edge  $v_3$  and the plane  $abc$  intersect (Figure 3). Obviously,  $v'c$  is parallel to the edge  $v_2$ . Since  $v'c$  does not intersect the interior of the triangle  $abc$  it follows that the vector  $\vec{v'c}$  lies “between” the vectors  $\vec{ba}$  and  $\vec{bc}$ . Therefore on the sphere the vertex  $v_2$  lies on the segment  $[t_1, t_2]$ .

Consider the plane  $\alpha$  of the third facet that contains the point  $c$  of the octahedron. It contains the line  $cv'$  and does not intersect the octahedron. This means that  $\alpha$  lies “between” the planes  $v'ca'$  and  $v'ca$  (that coincide with the plane  $abc$ ). The line  $v_2t_3$  (on the sphere) corresponds to the plane  $v'ca'$  and the line  $t_1v_2$  corresponds to the plane  $v'ca$ . Therefore the third facet corresponds to a line “between”  $v_2t_1$  and  $v_2t_3$ .

It is clear that the second facet corresponds to the line passing through  $t_1$  that lies “between”  $[t_1, t_2]$  and  $[t_1, t_3]$ . Therefore the point  $v_3$  lies inside the triangle  $t_1v_2t_3$ .

To prove the lemma in the other direction we note that for any triangle from the statement of the lemma it is possible to construct an inscribed octahedron by the way depicted in Figure 3.

Consider the second case. Without loss of generality we may assume that the first facet of  $A$  contains the edge  $ab$  of the octahedron, the second facet contains the edge  $ca'$ , and the third facet contains the edge  $b'c'$ . This means that extensions of the sides of the spherical triangle  $\Delta v_1v_2v_3$  pass through the vertices of  $\Delta t_1t_2t_3$  (see Figure 4).

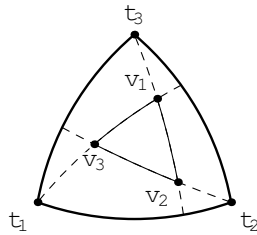


Fig. 4.

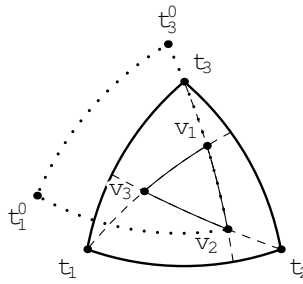


Fig. 5.

Let us show that  $\Delta v_1v_2v_3$  can be placed in  $\Delta t_1t_2t_3$  in the proper way. Since the area of  $\Delta v_1v_2v_3$  is less than the area of  $\Delta t_1t_2t_3$  it follows that one of the angles of  $\Delta v_1v_2v_3$  is less than  $\angle t_2t_1t_3$  (note that  $\Delta t_1t_2t_3$  is regular). Without loss of generality we may assume that this angle is  $\angle v_2$  and the triangle  $v_1v_2v_3$  is placed in  $t_1t_2t_3$  in the way shown in Figure 4 (the points  $t_1$  and  $v_3$  are on the same side of the line  $v_1v_2$ ).

Choose a point  $t_1'$  so that  $|t_1'v_2| = |t_1t_2|$  and  $\angle t_1'v_2t_3 = \angle t_1t_2t_3$ ; and choose a point  $t_3'$  on the ray  $[v_2, t_3)$  so that  $|t_3'v_2| = |t_1t_2|$  (Figure 5). Note that  $\angle t_3v_2t_1 > \angle t_3t_2t_1$ . Therefore the segment  $[t_1', v_2]$  intersects the segment  $[t_1, v_3]$ , thus giving the inclusion  $\Delta v_1v_2v_3 \subset \Delta t_1'v_1v_2$ . We obtain that  $\Delta v_1v_2v_3$  is positioned in the proper way inside  $\Delta v_2t_1't_3'$ , which is congruent to  $\Delta t_1t_2t_3$ .  $\square$

**Lemma 3.3.** *In Lemma 3.2 we may assume that  $|v_1v_2| \geq |v_1v_3| \geq |v_2v_3|$ .*

*Proof of Lemma 3.3.* If  $|v_1v_3| > |v_1v_2|$  then reflect  $\Delta v_1v_2v_3$  with respect to the bisector of  $\angle v_2v_1v_3$  and the triangle  $v_1v_2'v_3'$ , which lies in  $\Delta t_1t_2t_3$  in a proper way, because  $v_2' \in [v_1, v_3]$  and  $\Delta v_1v_2t_3 \subset v_1v_3't_3$ .

If  $|v_2v_3| > |v_1v_3|$  then reflect  $\Delta v_1v_2v_3$  with respect to the perpendicular bisector of the segment  $[v_1, v_2]$ . Denote by  $v_3'$  the image of  $v_3$ . We have

$$\angle v_3'v_2v_1 = \angle v_3v_1v_2 < \angle t_3v_1v_2 < \angle t_3v_2v_1.$$

Since  $|v_2v_3| > |v_1v_3|$  we have

$$\angle v_3'v_1v_2 = \angle v_3v_2v_1 < \angle v_3v_1v_2 < \angle t_3v_1v_2.$$

Therefore the “rays”  $[v_1, v'_3)$  and  $[v_2, v'_3)$  are directed into the interior of  $\triangle t_1 v_2 t_3$  and the point  $v'_3$  lies inside this triangle.

Using this two kinds of operations we can rearrange the side lengths of the  $\triangle v_1 v_2 v_3$  in the required order.  $\square$

**Corollary 3.4.** *If all facet angles of a solid angle  $A \in \mathcal{A}^3$  are less than  $\pi/6$  then  $A \in \mathcal{A}_S^3$ . If one facet angle of a solid angle  $A \in \mathcal{A}^3$  is greater than  $\pi/3$  then  $A \in \mathcal{A}^3 \setminus \mathcal{A}_S^3$ .*

Now we make the final step:

*Proof of Lemma 3.1.* Consider the triangle  $T$  corresponding to a solid angle  $A \in \mathcal{A}^3 \setminus \mathcal{A}_S^3$ . Let  $T = \triangle v_1 v_2 v_3$  and  $|v_1 v_2| \geq |v_1 v_3| \geq |v_2 v_3|$ . Let  $T_0$  be the regular triangle  $t_1 t_2 t_3$  with sides length  $\pi/3$ .

We are going to show how to increase the sides of the triangle  $T$  and obtain a triangle with a side greater than  $\pi/3$  (the set of triangles of this kind is obviously arcwise connected and by Corollary 3.4 belongs to the set  $\mathcal{A}^3 \setminus \mathcal{A}_S^3$ ).

Suppose all sides are less than  $\pi/3$ . Let us try to place the triangle  $T$  into the right triangle  $T_0$  in the way prescribed by Lemma 3.3. The only way that makes the position not proper is that  $v_3$  is outside  $\triangle v_1 v_2 t_3$ , which is possible only if the segment  $v_1 v_3$  goes outside the segment  $v_2 t_3$ . From Corollary 3.4 it follows that  $|v_1 v_2| \geq \pi/6$  and therefore  $\angle v_1 v_2 t_3$  is acute. Now we increase the length of the side  $v_1 v_3$  up to  $|v_1 v_2|$  preserving the angle  $\angle v_2 v_1 v_3$ . The position of  $\triangle v_1 v_2 v_3$  remains not proper (as required by Lemma 3.3).

Now we start to increase the length of the sides  $v_1 v_2$  and  $v_1 v_3$  in such a way that they remain equal during the process. The angle  $\angle v_1 v_2 v_3$  will increase while the angle  $\angle v_1 v_2 t_3$  will decrease during this process. Hence the point  $v_3$  will remain outside  $\triangle v_1 v_2 t_3$ . Finally we obtain an isosceles triangle with two sides greater than  $\pi/3$ .

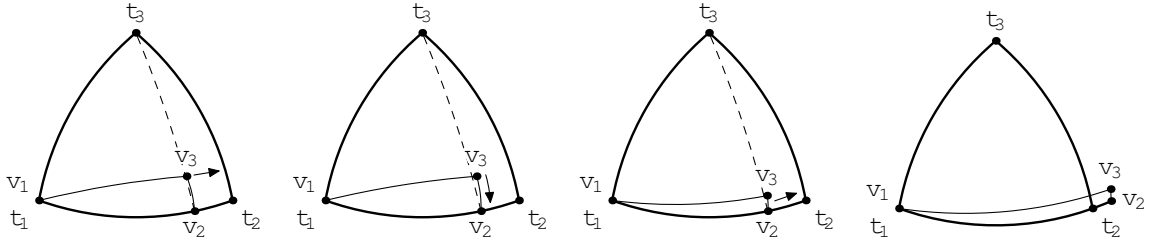


Fig. 6

$\square$

#### 4. THE CASE OF NON-SIMPLE POLYTOPES

In this case we have a weaker analogue of Lemma 3.2. We again associate a solid angle  $A$  with its spheric convex polygon and denote  $T_0$  the regular spherical triangle with side length  $\pi/3$ .

**Lemma 4.1.** *If  $A \in \mathcal{A} \setminus \mathcal{A}_S$  then some congruence takes the spherical polygon of  $A$  inside  $T_0$ .*

*Proof.* Consider a regular tetrahedron  $\Theta$  formed by some four facets of  $C$ . Look at  $\Theta$  from the vertex of  $A$  (it cannot be inside  $\Theta$ ), there are two alternatives:

Case I: We see some vertex of  $\Theta$ . Denote by  $B$  the solid angle of this vertex, its spherical triangle is congruent to  $T_0$ . Let us make a small perturbation of  $A$  keeping  $C$  inscribed into  $X$  and making the intersection  $\partial A \cap \partial B$  transversal.

Let  $\partial B$  consists of three flat angles  $B_1$ ,  $B_2$ , and  $B_3$ . Every intersection  $X_i = A \cap B_i$  is a convex set containing the vertex  $s$  of  $B_i$  and having three vertices  $x_1$ ,  $x_2$ , and  $x_3$  of the

octahedron  $C$  on its boundary. Consider the facet  $B_1$ , in this case  $x_1$ ,  $x_2$  and  $x_3$  is the triple of vertices  $a$ ,  $b'$  and  $c$  (Figure 7)

Note that  $a$  and  $c$  are on the sides of  $B_1$  and the segments  $[a, b']$  and  $[b', c]$  are parallel to the sides of  $B_1$ , so  $sab'c$  is a parallelogram. There exists a support line  $\ell_1$  to  $X_1$  passing through  $b'$  in the plane of  $B_1$ . The points  $a$ ,  $c$ , and  $s$  are on the one side of  $\ell_1$  and the line  $\ell_1$  separates  $X_1$  from infinity, except for the case when  $\ell_1$  is parallel to a side of  $B_1$ . But the latter situation is degenerate and can be excluded by a small perturbation. Analogous statement holds for facets  $B_2$  and  $B_3$ .

Thus  $\partial A \cap \partial B$  is bounded and after the translation that identifies the vertices of  $A$  and  $B$  the whole solid angle  $A$  will get inside  $B$ . This is true after an arbitrarily small perturbation of  $A$ , so its is was true for the original  $A$  by the continuity.

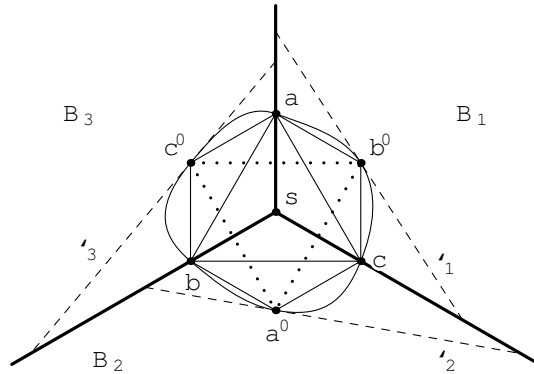


Fig. 7.

Case II: We see all vertices of  $\Theta$ . But in this case we do not see the vertex of  $C$  that corresponds to the farthest in the pair of edges of  $\Theta$  that intersect as we see from the vertex of  $A$ .  $\square$

It makes sense to make a definition:

**Definition 4.2.** Denote  $\mathcal{A}_0 \subset \mathcal{A}$  the set of solid angles that cannot be put into  $T_0$  by a congruence.

*Remark 4.3.* A careful analysis of Lemmas 3.2 and 3.3 shows that  $\mathcal{A}^3 \cap \mathcal{A}_0 \neq \mathcal{A}^3 \setminus \mathcal{A}_S^3$ . It is sufficient to take  $T = T_0$  and shrink one of its sides to the midpoint of that side slightly.

**Theorem 4.4.** Suppose  $P$  is a polytope in  $\mathbb{R}^3$  such that all its solid angles either in  $\mathcal{A}^3$ , or not in  $\mathcal{A}_0$ . Then there exists a regular octahedron inscribed into  $\partial P$ .

*Proof.* It remains to show that any  $A \in \mathcal{A}_0$  can be deformed to a halfspace inside  $\mathcal{A}_0$ . We can make a strong monotonic (monotonic with respect to inclusion) deformation of  $A$  to a halfspace, and obviously  $A$  will remain in  $\mathcal{A}_0$  under such deformation.  $\square$

## REFERENCES

- [1] A. Borel, J.C. Moore. Homology theory for locally compact spaces. // Michigan Mathematical Journal, 7 (1960), 137–159.
- [2] V.V. Makeev. Universally inscribed and outscribed polytopes. Doctor of Sciences thesis. Saint-Petersburg State University, 2003.
- [3] B. Matschke. On the square peg problem and some relatives. // arXiv:1001.0186, 2009.
- [4] R.N. Karasev. Inscribing a regular crosspolytope. // arXiv:0905.2671, 2009.
- [5] R.N. Karasev. Equipartition of a measure by  $(\mathbb{Z}_p)^k$ -invariant fans. // Discrete and Computational Geometry, 43:2 (2010), 477–481.
- [6] R.N. Karasev. Knaster's problem for  $(\mathbb{Z}_2)^k$ -symmetric subsets of the sphere  $S^{2^k-1}$ . // Discrete and Computational Geometry, 44:2 (2010), 429–438.

- [7] V. Klee, S. Wagon. Old and new unsolved problems in plane geometry and number theory. Dolciani Mathematical Expositions, The Mathematical Association of America, 1996.
- [8] I. Pak. The discrete square peg problem. // arXiv:0804.0657, 2008.
- [9] I. Pak. Lectures on discrete and polyhedral geometry. Preliminary version available at authors web page, 2010.
- [10] L.G. Schnirelmann. On some geometric properties of closed curves. (In Russian) // Uspehi Mat. Nauk, 10 (1944), 34–44.

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